LECTURE 3: BEILINSON'S CONJECTURES

In this lecture we shall finally state Beilinson's conjectures, which are higher-dimensional generalisations of the analytic class number formula. They are firstly of the form

"rank of a cycle group" = "order of vanishing of an L-function at an integer"

and then they pin down (up to a non-zero multiple in $\mathbb{Q}$) the residue of the L-function at that integer as the covolume of a cycle-theoretically defined lattice under a regulator. I think it is well worth dwelling for a few moments on how astounding that is; on the left hand side is a very mysterious and manifestly algebraic invariant of an algebraic variety, and on the right hand side is an (also very mysterious) analytic quantity.

The Beilinson conjectures are naturally broken down into three conjectures, depending on whether we are in the region of absolute convergence for the L-function, at a near-central point or at the central point. (There does exist a uniform statement, but it uses the language of mixed motives...) As we get closer to the central point, the intricacy of the conjecture becomes more involved.

Remark 0.1. We saw in Lecture 1 in the discussion of the Dirichlet regulator that we needed to consider not $H^1_{\mathcal{M}}(F, \mathbb{Q}(1)) = F^* \otimes_{\mathbb{Z}} \mathbb{Q}$, but rather the units $H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(1)) = \mathcal{O}_F^* \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that when we defined motivic cohomology we never really used the base field $\operatorname{Spec} k$; the definition works equally well for schemes of finite type over $\operatorname{Spec} \mathbb{Z}$ (motivic cohomology is an "absolute" theory). For a variety X over $\operatorname{Spec} \mathbb{Q}$, we define

$$H^i_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(n)) := \operatorname{im}(H^i_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(n)) \to H^i_{\mathcal{M}}(X, \mathbb{Q}(n)))$$

where $\mathcal{X}$ is a proper regular model of X over $\operatorname{Spec} \mathbb{Z}$ (we assume that such a $\mathcal{X}$ exists). Then $H^i_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(n))$ is independent of the choice of model. Note that it is conjectured that $H^i_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(n)) = H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$ for $n > i$ (see the exercises).

Note that proper regular models are rarely known to exist, so one works with a more complicated definition using regular flat models and alterations instead.

1. Euler factors at ∞

Last time I discussed Euler factors at primes, but I ran out of time to talk about Euler factors at ∞ (the archimedean prime), so let's do that now. Recall that X is a smooth projective variety over $\mathbb{Q}$. Let $\Gamma(s)$ denote the gamma function (i.e. the usual analytic continuation of $n \mapsto (n-1)!$ from $\mathbb{N}$ to $\mathbb{C}$). Set

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$$
$$\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)\,.$$

Recall the Hodge decomposition

$$H^i_{\mathrm{sing}}(X_{\mathbb{C}}, \mathbb{C}) \cong \bigoplus_{\substack{p+q=i \\ p,q\geq 0}} H^q(X_{\mathbb{C}}, \Omega^p_{X_{\mathbb{C}}})\,.$$

Put $h^{p,q} := \dim_{\mathbb{C}} H^q(X_{\mathbb{C}}, \Omega^p_{X_{\mathbb{C}}})$ and $h^{p,\pm} := \dim_{\mathbb{C}} H^{p,\pm(-1)^p}$ where

$$H^p(X_{\mathbb{C}}, \Omega^p_{X_{\mathbb{C}}}) = H^{p,+} \oplus H^{p,-}$$

is the decomposition into eigenspaces with respect to the involution induced by complex conjugation on $X_{\mathbb{C}}$.

Definition 1.1. The Euler factor of $H^i(X)$ at ∞ is

$$L_\infty(H^i(X), s) = \begin{cases} \prod_{\substack{p+q=i \\ p<q}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} & \text{if } i \text{ is odd} \\ \prod_{\substack{p+q=i \\ p<q}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \Gamma_{\mathbb{R}}(s-i/2)^{h^{i/2,+}} \cdot \Gamma_{\mathbb{R}}(s-i/2+1)^{h^{i/2,-}} & \text{if } i \text{ is even}. \end{cases}$$

Definition 1.2. The completed L-function of $H^i(X)$ is

$$\tilde{L}(H^i(X), s) := L(H^i(X), s) L_\infty(H^i(X), s).$$

This might look a little unmotivated, but one could look at the special case of the Riemann zeta function. There one analytically continues $\Gamma(\frac{s}{2})\zeta(s)$ by writing it as a Mellin transform (and of course the Gamma function is defined as the Mellin transform of e^{-x}). Tate's thesis provides a way of treating the Euler factors of zeta functions, both at primes and at infinity, in a uniform way.

Conjecture 1.3 (Serre). $L(H^i(X), s)$ converges absolutely for $\mathrm{Re} > \frac{i}{2}+1$ (and hence does not vanish in that region). It has a meromorphic continuation to $\mathbb{C}$ with the only possible poles at $s = \frac{i}{2}+1$ when i is even. The completed L-function satisfies a functional equation

$$\tilde{L}(H^i(X), s) = \epsilon(H^i(X), s)\tilde{L}(H^i(X), i+1-s)$$

where $\epsilon(H^i(X), s)$ is a non-zero holomorphic function.

Serre's conjecture is known in very few examples. We have seen it for $X = \operatorname{Spec} F$ in Lecture 1. It is known for modular curves, elliptic curves over $\mathbb{Q}$ (by Wiles), CM abelian varieties and a few other cases. The (only?) strategy to prove this is to show that $L(H^i(X), s)$ is "modular", i.e. that it is the L-function of some automorphic object where meromorphic continuation is known. This was one of the motivations behind Langlands' original work on the Langlands programme, I believe.

2. Deligne cohomology and order of vanishing of L-functions

It follows quite immediately from the definition of $L_\infty(H^i(X), s)$ and the fact that the gamma function has simple poles precisely at $-1, -2, -3, \ldots$ that

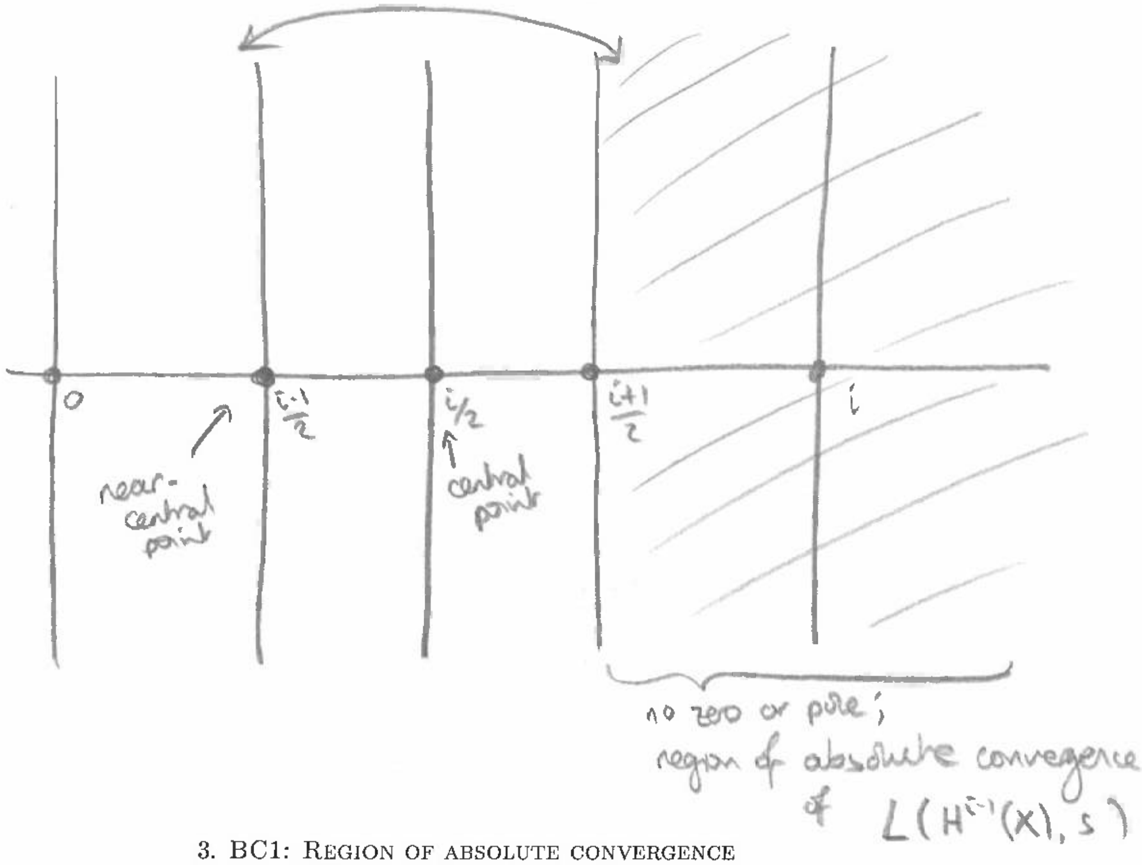
$$\dim_{\mathbb{R}} H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) = -\mathrm{ord}_{s=i-n} L_\infty(H^{i-1}(X), s)$$

for all i, n with $i < 2n$. Since $L(H^{i-1}(X), s)$ has no zero or pole in the region $\mathrm{Re}(s) > (i+1)/2$ (we assume all conjectures from Lecture 2, of course!), the functional equation $s \leftrightsquigarrow i - s$ for $\tilde{L}(H^{i-1}(X), s)$ yields

$$\dim_{\mathbb{R}} H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) = \mathrm{ord}_{s=i-n} L(H^{i-1}(X), s) \tag{2.0.1}$$

if $i + 1 < 2n$.

Let $j = i - n$. Then we see that (2.0.1) happens precisely when $j \neq \frac{i}{2}, \frac{i-1}{2}$. That is, when j is not the *central point* $\frac{i}{2}$ of the functional equation (for $H^{i-1}(X)$) or the *near-central point* $\frac{i-1}{2}$.



3. BC1: Region of absolute convergence

Let X be a smooth projective variety over $\mathbb{Q}$. Let m be an integer in the right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{i+1}{2}\}$ where $L(H^{i-1}(X), s)$ absolutely converges. Consider the Beilinson regulator

$$r_{\mathcal{B}} = r_{\mathcal{B}}^{m,i} : H^i_{\mathcal{M}}(X, \mathbb{Q}(m)) \to H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(m)) \simeq \mathbb{R}^{d_{m,i}}$$

where $d_{m,i} = d_{m,i}(X) = \mathrm{ord}_{s=i-m} L(H^{i-1}(X), s)$.

For $m \geq \frac{i+1}{2}$ the Deligne cohomology sits in a short exact sequence

$$0 \to \mathrm{Fil}^m H^{i-1}_{\mathrm{dR}}(X_{\mathbb{R}}) \to H^{i-1}_{\mathrm{sing}}(X_{\mathbb{R}}, \mathbb{R}(m-1)) \to H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(m)) \to 0\,.$$

Consider the $\mathbb{Q}$-structure on $\det H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(m))$ given by

$$\Lambda_{m,i} := \det H^{i-1}_{\mathrm{sing}}(X_{\mathbb{R}}, \mathbb{Q}(m-1)) \otimes (\det \mathrm{Fil}^m H^{i-1}_{\mathrm{dR}}(X))^{\vee}\,.$$

(Recall that det means the top exterior power of a finite dimensional vector space, and $\vee$ means the dual). For any $\mathbb{Q}$-structure V of $H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(m))$, we define the volume $\mathrm{vol}(V)$ of V by

$$\det V = \mathrm{vol}(V) \cdot \Lambda_{m,i}\,.$$

Conjecture 3.1 (BC1). Let X be a smooth projective variety over $\mathbb{Q}$. Let $m > \frac{i+1}{2}$. Consider the Beilinson regulator on the integral part of motivic cohomology:

$$r_{\mathcal{B}} : H^i_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(m)) \to H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(m)) \simeq \mathbb{R}^{d_{m,i}}\,.$$

Then:

a) $r_{\mathcal{B}}(H^i_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(m)))$ is a $\mathbb{Q}$-structure on $H^i_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(m))$.
b) $\ker(r_{\mathcal{B}})=0$.
c) $\mathrm{vol}(H^i_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(m))) \sim_{\mathbb{Q}^*} \lim_{s\to i-m}(s-i+m)^{-d_{m,i}}L(H^{i-1}(X),s)$.

Remarks 3.2. (1) An immediate corollary of the conjecture is that for integers $m>\frac{i+1}{2}$ we have

$$\dim_{\mathbb{Q}} H^i_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(m)) = \dim_{\mathbb{Q}} H^{i-1}_{\mathrm{sing}}(X_{\mathbb{R}},\mathbb{Q}(m-1)) - \dim_{\mathbb{Q}} \mathrm{Fil}^m H^{i-1}_{\mathrm{dR}}(X)\,.$$

(2) When $m\geq i-1$ then there is no Hodge filtration part, so everything is given in terms of the rational Betti/singular cohomology.

4. BC2: The near-central point

Let X be a smooth projective variety over $\mathbb{Q}$. Let us consider what happens at the near-central point for $L(H^{i-1}(X),s)$, i.e. at $m=\frac{i+1}{2}$. Re-arranging, we are looking at the regulator

$$r_{\mathcal{B}} = r_{\mathcal{B}}^{2j-1,j} : H^{2j-1}_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(j)) \to H^{2j-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))\,.$$

We have already seen the first case $j=1$ for $X=\operatorname{Spec}F$, F a number field. Indeed, we saw that

$$\mathcal{O}_F^*\otimes\mathbb{Q} \cong H^1_{\mathcal{M}}(\mathcal{O}_F,\mathbb{Q}(1)) \xrightarrow{r_{\mathcal{B}}} H^1_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(1)) \simeq \mathbb{R}^{r_1+r_2}$$

is the Dirichlet regulator. Recall from Lecture 1 that the image of $\mathcal{O}_F^*\otimes\mathbb{Q}$ is a full rank lattice in $\mathbb{R}^{r_1+r_2-1}$. In light of this example, consider then the "thickened" regulator

$$\tilde{r}_{\mathcal{B}} : H^{2j-1}_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(j)) \oplus N^{j-1}(X)_{\mathbb{Q}} \to H^{2j-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))$$

where $N^{j-1}(X)=Z^{j-1}(X)/\sim_{\mathrm{hom}}$ is the group of codimension $j-1$ cycles on X modulo homological equivalence (see the exercise sheet). $\tilde{r}_{\mathcal{B}} := r_{\mathcal{B}}\oplus \mathrm{cl}_{\mathrm{dR}}$ where

$$\mathrm{cl}_{\mathrm{dR}} : N^{j-1}(X) \to H^{2j-2}_{\mathrm{dR}}(X_{\mathbb{R}})\,.$$

is (induced by) the de Rham cycle class map. Note that $N^0(\operatorname{Spec}F)=\mathbb{Z}$.

Conjecture 4.1 (BC2). Let X be a smooth projective variety over $\mathbb{Q}$. Consider the thickened Beilinson regulator

$$\tilde{r}_{\mathcal{B}} : H^{2j-1}_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(j)) \oplus N^{j-1}(X)_{\mathbb{Q}} \to H^{2j-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))\,.$$

Then:
a) $\tilde{r}_{\mathcal{B}}(H^{2j-1}_{\mathcal{M}/\mathbb{Z}}(X,\mathbb{Q}(j)) \oplus N^{j-1}(X)_{\mathbb{Q}})$ is a $\mathbb{Q}$-structure on $H^{2j-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))$.
b) $\ker(\tilde{r}_{\mathcal{B}})=0$.
c) $\mathrm{vol}(\mathrm{im}(\tilde{r}_{\mathcal{B}})) \sim_{\mathbb{Q}^*} \lim_{s\to j}(s-j)^{-d_{j-1,2j-1}}L(H^{2j-2}(X),s)$.

Remark 4.2. See the exercise sheet for the relationship with the Tate conjecture.

5. BC3: The central point

Let X be a smooth projective variety over $\mathbb{Q}$. Let us consider what happens at the near-central point for $L(H^{i-1}(X),s)$, i.e. at $m=\frac{i}{2}$. In this case the target of the Beilinson regulator vanishes, so formulating Beilinson's conjecture is trickier. Let us look at a famous example for $m=1$ to get hints at what the conjecture should be:

5.1. **The Birch and Swinnerton-Dyer conjecture.** Let E be an elliptic curve over $\mathbb{Q}$. The Mordell-Weil theorem says that the group of rational points $E(\mathbb{Q})$ is finitely generated. Work of Mazur has determined all the possible groups occurring as $E(\mathbb{Q})_{\mathrm{tors}}$, but the rank of $E(\mathbb{Q})$ is very mysterious. In fact, the first part of the Birch and Swinnerton-Dyer conjecture asserts that

$$\operatorname{rank} E(\mathbb{Q}) = \operatorname{ord}_{s=1} L(H^1(E), s).$$

The second part of the BSD conjecture pins down the residue at $s = 1$:

$$\lim_{s\to 1}(s-1)^{-\operatorname{rank} E(\mathbb{Q})} L(H^1(E), s) = \frac{|\text{Ш}(E/\mathbb{Q})| \cdot \prod_p c_p}{|E(\mathbb{Q})_{\mathrm{tors}}|^2} \cdot \Omega_E \cdot \det(\langle -,- \rangle).$$

Don't worry if you don't know what the terms in the fraction mean; (conjecturally on the finiteness of $\text{Ш}(E/\mathbb{Q})$; itself part of the conjecture) the fraction is a rational number so we ignore it for Beilinson conjecture purposes. The more important quantities for us are the number $\Omega_E := \int_{E(\mathbb{R})} \omega$ is the period of E (ω is the invariant differential), and

$$\langle -,- \rangle \,:\, E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$

is the height pairing. We shall need higher analogues of each to formulate Beilinson's conjecture at the central point.

Let $\mathrm{CH}^n(X) = H^{2n}_{\mathcal{M}}(X, \mathbb{Z}(n))$ be the Chow group of codimension-n cycles on X (i.e. $\mathrm{CH}^n(X) := Z^n(X)/\sim_{\mathrm{rat}}$. Let $\mathrm{CH}^n(X)_0 := \ker(\mathrm{cl} : \mathrm{CH}^n(X) \to H^{2n}(X(\mathbb{C}), \mathbb{C}))$ be the subgroup of cycles which are homologically equivalent to zero.

Conjecture 5.2 (Beilinson, Bloch). There exists a natural non-degenerate height pairing

$$\langle -,- \rangle_n \,:\, \mathrm{CH}^n(X)_0 \otimes \mathbb{Q} \times \mathrm{CH}^{\dim X - n+1}(X)_0 \otimes \mathbb{Q} \to \mathbb{R}.$$

There are proposals for a construction of this height pairing, but they all rely on conjectures unless $n = 1$. The $n = 1$ case reduces to the Néron-Tate height pairing of the Picard variety of X via $\mathrm{CH}^1(X)_0 \simeq \mathrm{Pic}^0(X)$.

Fortunately the higher-dimensional generalisation of Ω_E is not conjectural. Consider the isomorphism

$$\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_{\mathbb{R}}) \xrightarrow{\sim} H^{2j-1}_{\mathrm{sing}}(X_{\mathbb{R}}, \mathbb{R}(j-1)).$$

The left hand side has the $\mathbb{Q}$-structure $\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X)$. We define the Deligne period of X to be

$$\Omega_{X,j} := \mathrm{vol}(\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X)).$$

Notice that $\Omega_{E,1} \sim_{\mathbb{Q}^*} \Omega_E$ for an elliptic curve E over $\mathbb{Q}$ because the invariant differential ω is a generator of $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(E_{\mathbb{R}})$ and the comparison isomorphism $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_{\mathbb{R}}) \xrightarrow{\sim} H^1_{\mathrm{sing}}(X_{\mathbb{R}}, \mathbb{R})$ sends ω to $\int_{E(\mathbb{R})} \omega$.

Conjecture 5.3 (BC3). Let X be a smooth projective variety over $\mathbb{Q}$. Then:

a) $\mathrm{CH}^j_{\mathbb{Z}}(X) := H^{2j}_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Z}(j))$ is finitely generated.
b) $\rho_j := \operatorname{rank} \mathrm{CH}^j_{\mathbb{Z}}(X) = \operatorname{ord}_{s=j} L(H^{2j-1}(X), s)$.
c) Let $\langle -,- \rangle_{j,\mathbb{Z}}$ denote the restriction of the height pairing $\langle -,- \rangle_j$ to $\mathrm{CH}^j_{\mathbb{Z}}(X)_0 \times \mathrm{CH}^{\dim X - j+}_{\mathbb{Z}}(X)_0$. Then

$$\lim_{s\to j}(s-j)^{-\rho_j} L(H^{2j-1}(X), s) \sim_{\mathbb{Q}^*} \Omega_{X,j} \cdot \det(\langle -,- \rangle_{j,\mathbb{Z}}).$$

By the previous discussion, BC3 specialises to the BSD conjecture when $X = E$ is an elliptic curve and $j = 1$ (but the BSD conjecture is precise about the rational factor).

6. Reflection

At this point it perhaps worth reflecting a little. The Beilinson conjectures BC1, BC2 and BC3 are conjectures about conjectural objects (the L-function, for example!), and are built on top of other conjectures. Moreover, they include extremely difficult open problems as special cases. But what do they actually say? They predict a link between between a very geometric object - a motivic cohomology group having to do with algebraic cycles - and the special values of an L-function. Both sides of the story are mysterious so the conjecture can be looked at from left to right or vice versa and be equally interesting.